Statistical Theory of Compound-Nuclear Reactions

Overview of theoretical developments since the 1950s

Hans A. Weidenmüller
MPI für Kernphysik
Heidelberg, Germany

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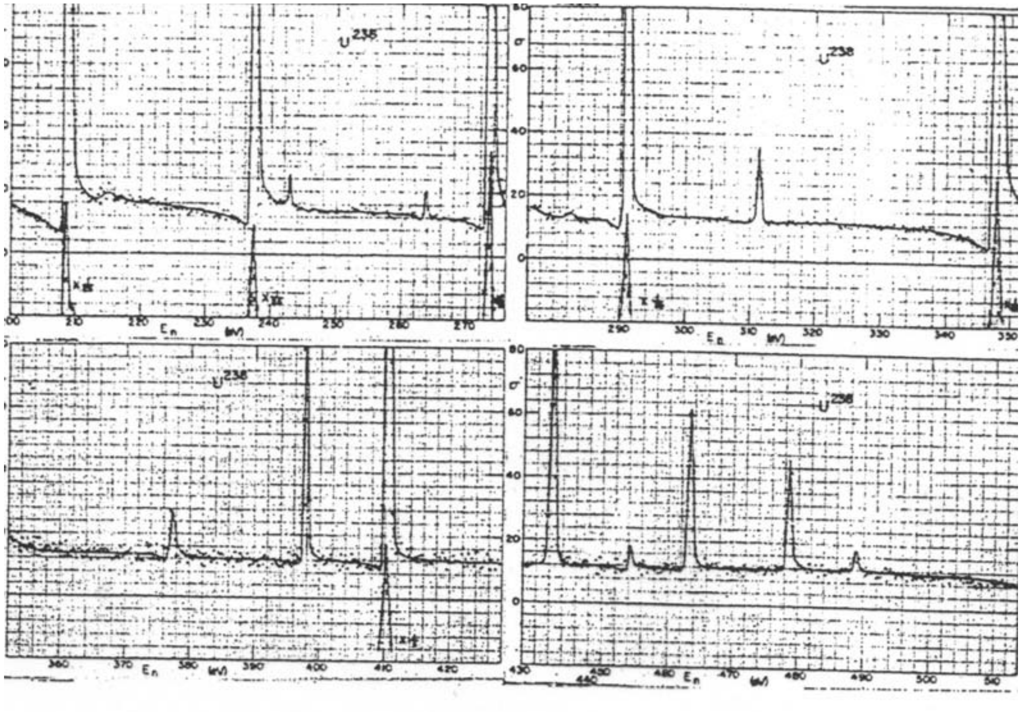
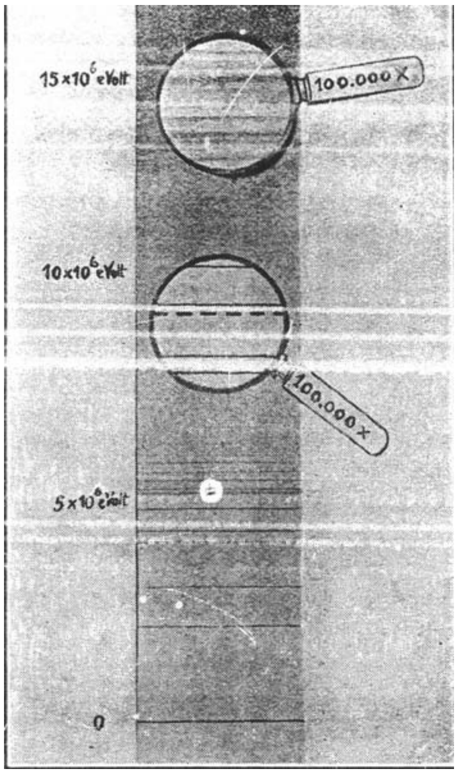
Contents:

1. Why a Statistical Theory?

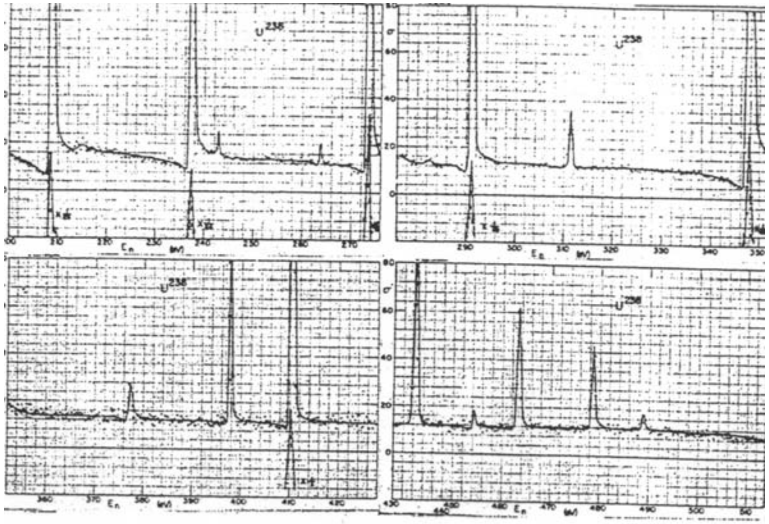
In medium-weight and heavy nuclei, neutron time-of-flight and proton scattering data display sequences of narrow and narrowly spaced resonances of the same spin and parity.



E. Fermi et al., Proc. Roy. Soc. A 146 (1934) 483,
ibid. 149 (1935) 522
J. B. Garg et al., Phys. Rev. 134 (1964) B 985
W. M. Wilson et al., Nucl. Phys. A 245 (1975) 285

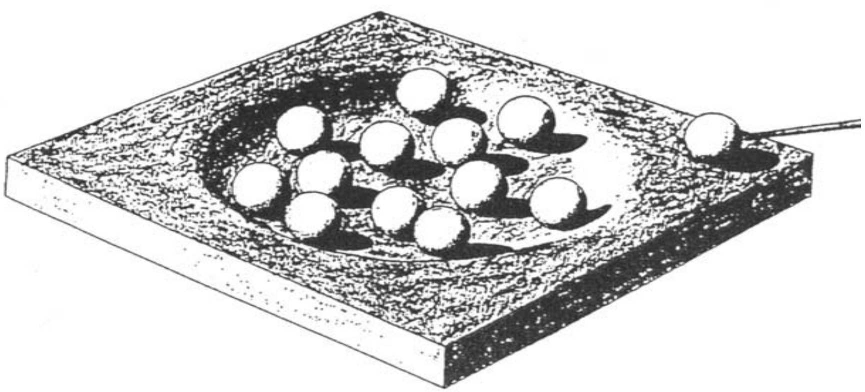


N. Bohr, Nature 137 (1936) 344.



Garg et al., Phys. Rev. 134 (1964) B 985

Niels Bohr: Numerous narrow resonances are incompatible with an independent-particle model. In the compound nucleus, many nucleons interact strongly! The nucleus equilibrates before it decays. Formation and decay of the compound nucleus are independent processes.



Nature 137 (1936) 351

Bohr's compound-nucleus idea formalized in the Hauser-Feshbach formula for the *average* reaction cross section.

Energy average (brackets) over many resonances. Open channels a,b,c specified by fragmentation and internal states of fragments (labels α, β), by angular momentum of relative motion, by total spin J. Differential cross section is bilinear in S(J). Assume $\langle S(J)S^*(J')\rangle = 0$ for $J \neq J'$ and that $\langle |S(J)_{ab}|^2 \rangle$ factorizes.

$$< d\sigma_{\alpha\beta} / d\Omega > = \sum \text{ (coefficients) } (T_a\, T_b / \sum T_c)\, P_l (\cos \theta) \,.$$

W. Hauser and H. Feshbach, Phys. Rev. 87 (1952) 366

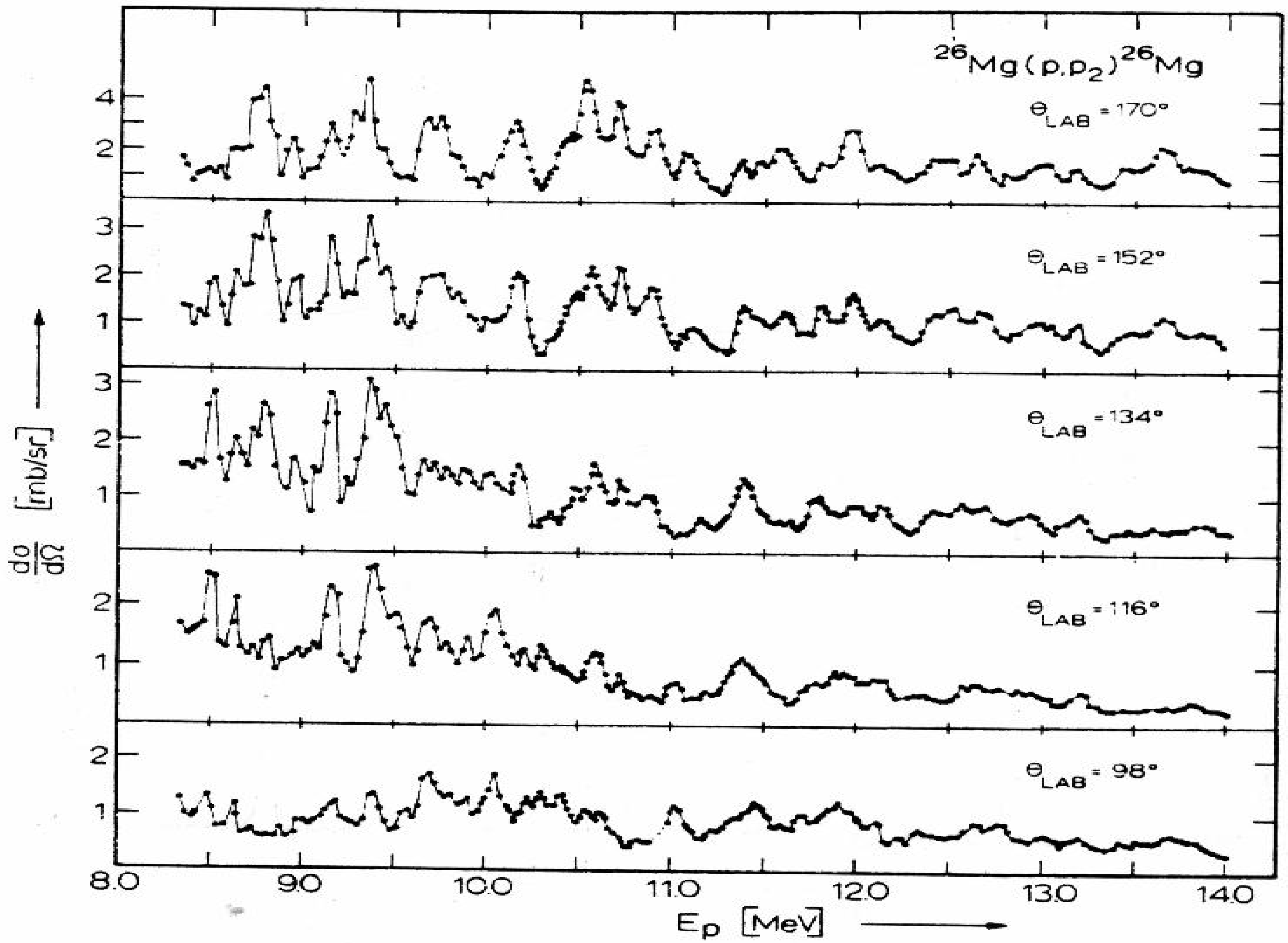
And there are statistical cross section fluctuations (Ericson).

With increasing excitation energy, average nuclear level density increases, average spacing d between resonances decreases. At the same time, total resonance width Γ increases (ever more decay channels open up). Transition from isolated resonances ($d >> \Gamma$) to weakly overlapping resonances ($\Gamma \approx d$) to strongly overlapping resonances ($\Gamma >> d$).

For $\Gamma >> d$ "Ericson fluctuations" predicted.

T. Ericson, Phys. Rev. Lett. 5 (1960) 430 and Ann. Phys. (N.Y.) 23 (1963) 390
D. Brink and R. O. Stephens, Phys. Lett. 5 (1963) 77

Such fluctuations were actually found experimentally several years later:



O. Häusser, A. Richter, W. von Witsch, W. J. Thompson, Nucl. Phys. A 109 (1968) 329

Task of theory:

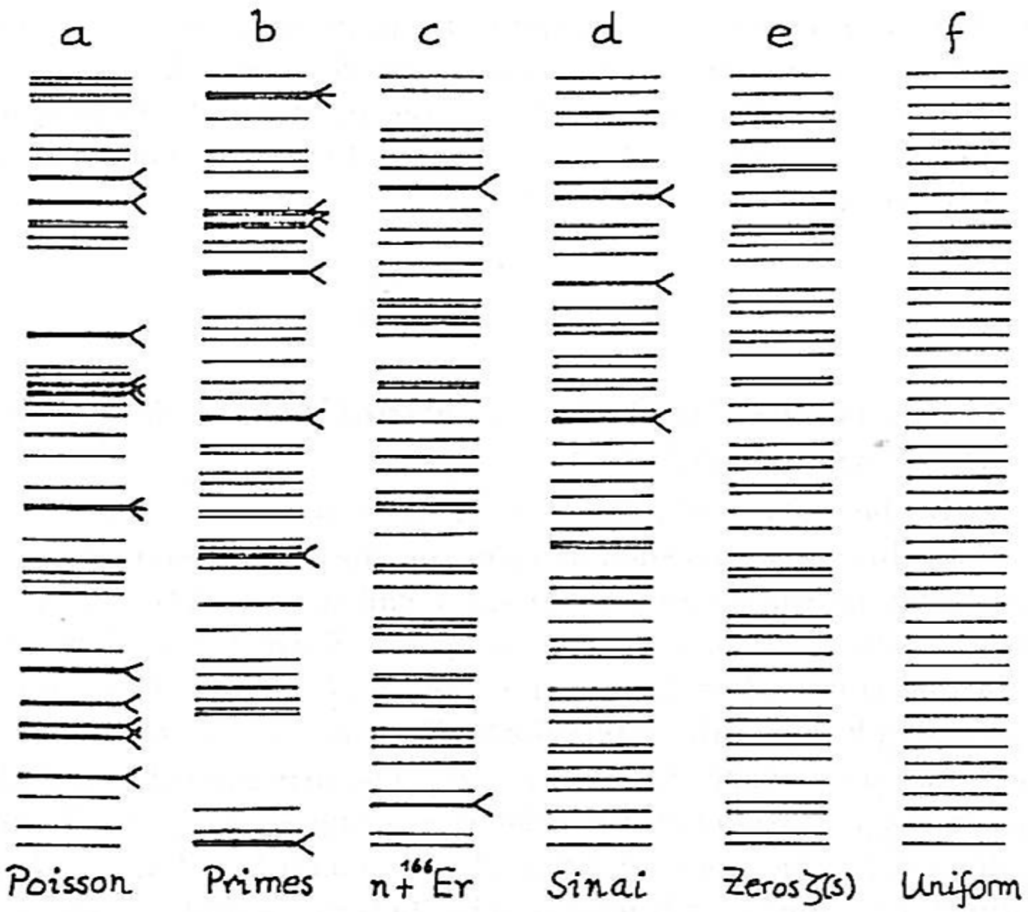
(i) Derive properties of average cross section and of cross-section fluctuations from dynamical theory of resonances.

(ii) Establish limits of validity, search for generalizations (preequilibrium reactions). Include direct reactions and symmetry violation.

Shell model is useless: Dimension of matrices would be about 10^6, required accuracy of matrix elements of residual interaction and required accuracy of numerical calculation (about 10 eV) are unattainable in the forseeable future.

Another starting point is needed: Random-matrix theory (RMT) describes spectral fluctuation properties of resonances correctly.

Spectral Fluctuations:



O. Bohigas and M. J. Giannoni, Lecture Notes in Physics 209 (1984), Springer-Verlag, Heidelberg

Random Matrix Theory (Wigner).

A theory that makes generic statements about eigenvalues and eigenfunctions of an arbitrary Hamiltonian matrix. Will here be applied to the resonances seen in slow neutron and proton scattering.

Consider matrix representation $H_{\mu\nu}$ of Hamiltonian in Hilbert space, with $\mu, \nu = 1, \ldots N$ and $N \gg 1$. Time-reversal invariance implies $H_{\mu\nu} = H_{\nu\mu}$ real. No further symmetries.

Take ensemble of such Hamiltonians. Should not have preferred direction in Hilbert space: Invariant under orthogonal transformations. "Gaussian orthogonal ensemble" (GOE).

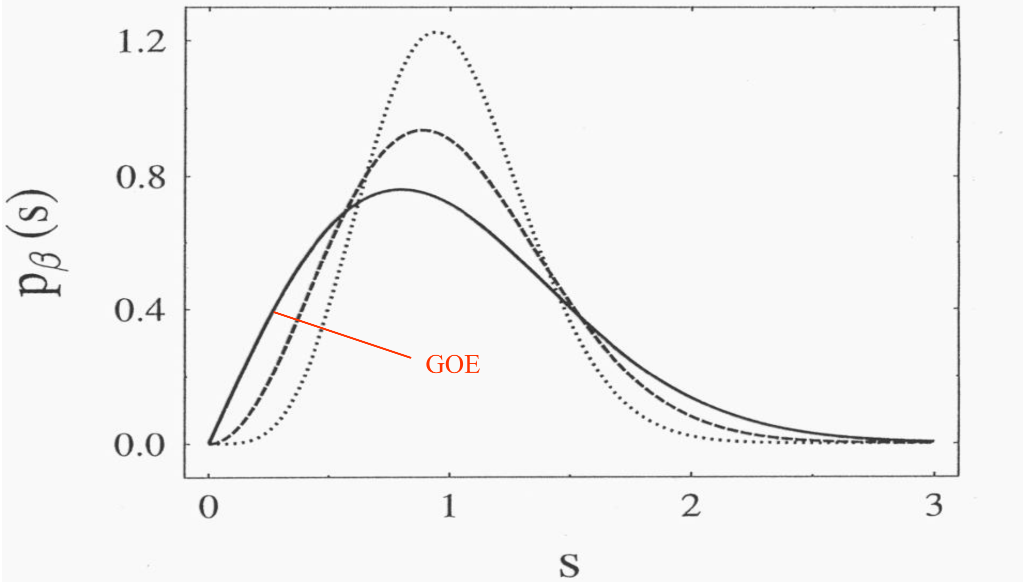
$$\mathcal{N} \exp [- (N / \lambda^2) \operatorname{Trace} (H^2)] \Pi_{\mu \leq \nu} \, d H_{\mu\nu}$$

Every state interacts with every other state. Single parameter lambda determines mean level density. Gaussian cutoff for convenience but results very general: Universality and ergodicity. Quantitative and parameter-free predictions possible for local spectral fluctuation measures.

Quantitative Predictions: $(N \to \infty)$

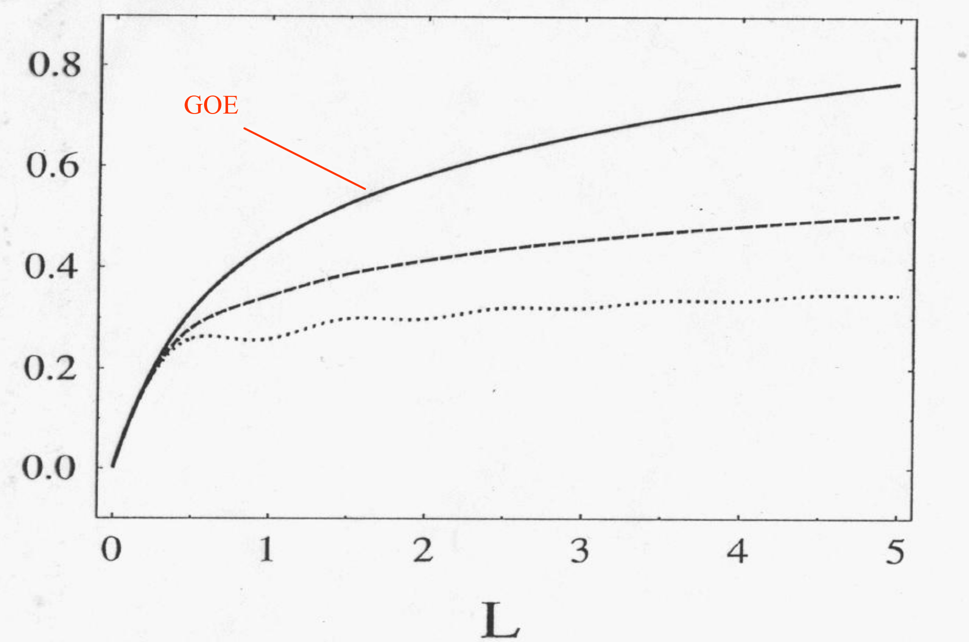
(a) Distribution of spacings of neighbouring levels (nearest-neighbor spacing distribution)

s is the level spacing in units of the average level spacing. Note the level repulsion at small spacings.



(b) Variance of the number of levels in an interval of length L

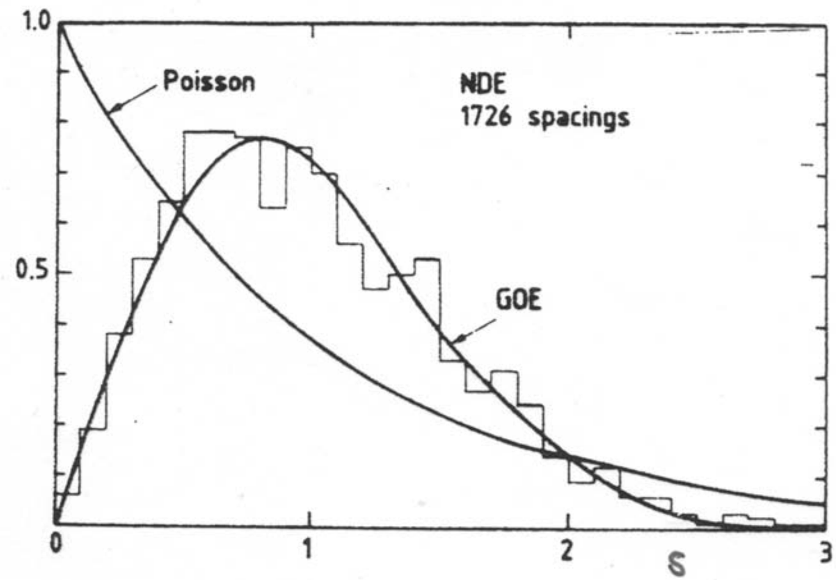
L is measured in units of the mean level spacing. The variance grows only logarithmically with L. Below, we use another related measure ("Dyson-Mehta Statistic" or "Delta3-Statistic").



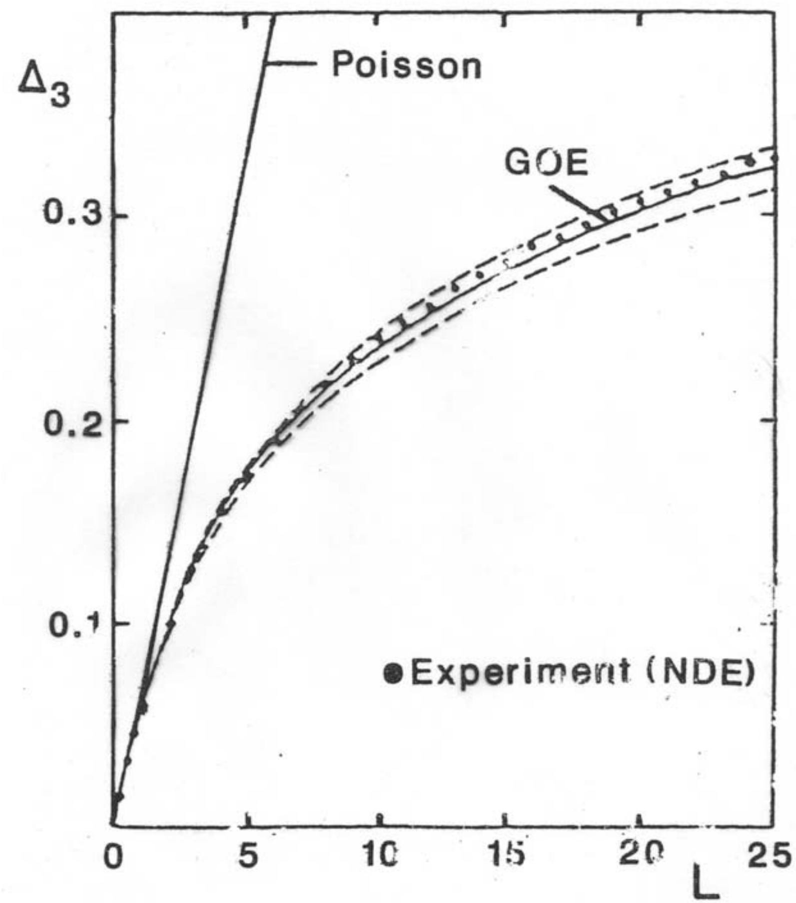
(c) Projections of eigenfunctions onto fixed vector in Hilbert space have Gaussian distribution

Evidence for GOE statistics in nuclei

Resonances near neutron threshold and near Coulomb barrier for protons.



R. Haq, A. Pandey, O. Bohigas,
Phys. Rev. Lett. 43 (1982) 1026
and Nuclear Data for Science
and Technology, Riedel (1983) 209.



There is also evidence supporting Gaussian distribution of eigenvectors ("Porter-Thomas distribution" of widths).

Program for a statistical theory:

Input: An ensemble of GOE Hamiltonians H^{GOE} for the resonances.
It replaces the actual nuclear Hamiltonian.

Theory cannot reproduce actual measured cross section but can only yield average cross section and cross section fluctuation (variance), both calculated as ensemble averages over the GOE.

Assume that resonances with different quantum numbers are described by uncorrelated GOE Hamiltonians. Then elements S(J) and S(J') of the scattering matrix for different spins $J \neq J'$ are uncorrelated, and compound-nucleus cross section is symmetric about 90 degrees c.m., in keeping with Hauser-Feshbach formula and with experimental evidence.

Task:

For each J express S(J) as function of H^{GOE}. Calculate ensemble averages $\langle S(J) \rangle$, variance $\langle |S(J)|^2 \rangle - |\langle S(J) \rangle|^2$, and fourth moment of S(J) as well as energy correlators.

From now on label J fixed and omitted.

2. Model for Resonance Reactions

Several formal theories of resonance reactions available.

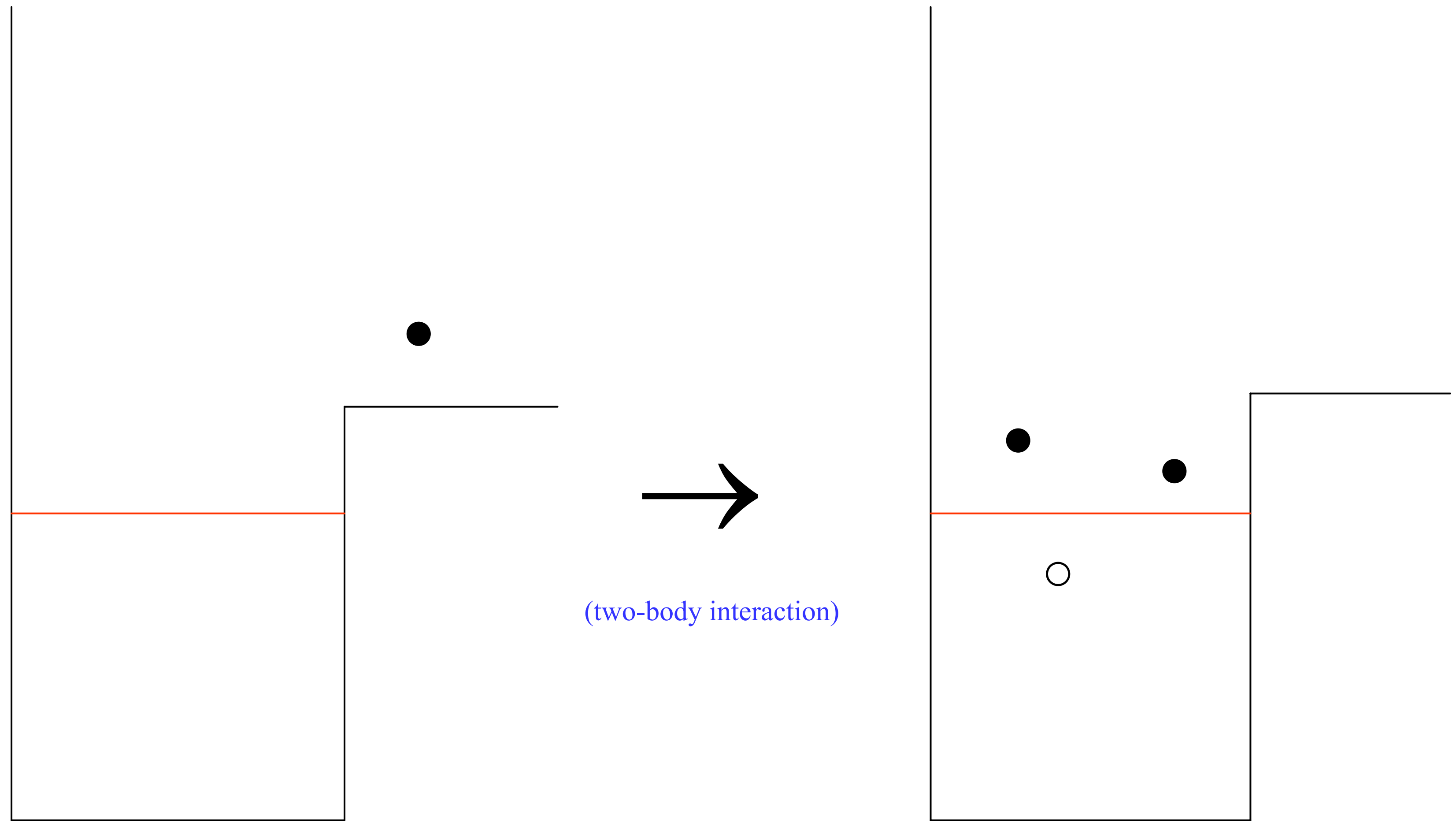
P. L. Kapur and R. Peierls, Proc. R. Soc. London A 166 (1938) 277
E. P. Wigner and L. Eisenbud, Phys. Rev. 72 (1947) 29
J. Humblet and L. Rosenfeld, Nucl. Phys. 26 (1961) 529

But we need a dynamical theory that relates Hamiltonian for the resonances and S-matrix. Two candidates.

H. Feshbach, Ann. Phys. (N.Y.) 5 (1958) 357 and 19 (1962) 287
C. Mahaux and H. A. Weidenmüller, Shell-Model Approach to Nuclear Reactions, North Holland, Amsterdam (1969)

Here we follow largely MW because exact expression for compound-nucleus cross section has been worked out in this framework. But other approaches were used, too, especially Humblet-Rosenfeld (Moldauer) and Feshbach.

Dynamical model for resonances (schematic): In shell model, a large number of quasibound states.



Scattering state

Quasibound state

Scattering states are dynamically coupled with each other.

That coupling results in non-resonant unitary background matrix $S^{(0)}_{ab}(E)$ with smooth (negligible) dependence on energy E. Contributes to direct reactions.

N >> 1 quasibound states {μ} are dynamically coupled with each other.
That coupling described by Hamiltonian matrix $H_{\mu\nu}$. (This only part of total Hamiltonian!)

Scattering states and quasibound states are dynamically coupled.
Complex matrix elements $X_{a\mu}$ with negligible energy dependence.

Without further specification of $H_{\mu\nu}$ this yields

Many-resonance formula for the scattering matrix:

$$S_{ab}(E) = S^{(0)}_{ab} - 2i\pi \sum_{\mu\nu} X_{a\mu} (D_{\mu\nu}(E))^{-1} X_{\nu b} \,,$$

$$D_{\mu\nu}(E) = E\, \delta_{\mu\nu} - H_{\mu\nu} + i\pi \sum_c X_{\mu c} X_{c\nu} \,.$$

Unitary S-matrix S_{ab} with N resonances and “width matrix” $2\pi \sum_c X_{\mu c} X_{c\nu}$.
“Effective Hamiltonian” $H_{\mu\nu} - i\pi \sum_c X_{\mu c} X_{c\nu}$.

Statistical Theory: Replace $H \to H^{GOE}$ and generate ensemble of S-matrices. Technical challenge: Calculate ensemble averages of S, of $|S|^2$, higher moments and energy correlation functions for all values of Γ / d. Average $\langle S_{ab} \rangle$ easy, higher moments not.

Identify theoretical ensemble averages with experimental energy averages.

3. Average S-matrix. Direct Reactions

Write $S_{ab}(E) = < S_{ab}(E) > + S^{fl}_{ab}(E)$ so that $<S^{fl}_{ab}(E)> = 0$ and

$$<|S_{ab}(E)|^2> = |<S_{ab}(E)>|^2 + <|S^{fl}_{ab}(E)|^2> .$$

Physical interpretation: $<S_{ab}(E)>$ (energy average!) describes *fast* part of reaction (uncertainty principle); $S^{fl}_{ab}(E)$ describes *slow* part (decay of the long-lived compound nucleus).

Simple models involving only few degrees of freedom can be used to calculate $<S_{a\,b}(E)>$: Optical model of elastic scattering for diagonal part, DWBA or coupled channels approach for non-diagonal part. Assume $<S_{ab}(E)>$ to be known, to be independent of energy. $<S_{ab}(E)> \rightarrow <S_{ab}>$ serves as input of statistical theory.

Statistical Theory: Predict $< | S^{fl}_{ab}(E)|^2>$, higher moments, and energy correlators from $<S_{ab}>$.

RMT always predicts fluctuations in terms of mean values (spectral fluctuations in terms of the average level spacing d, for instance).

Unitarity deficit of $\langle S_{ab} \rangle$: $\sum_b |\langle S_{ab} \rangle|^2 < 1$.

Easy to see for the single-channel case. Imaginary part of optical model describes loss of scattering amplitude due to compound-nucleus formation. Imaginary part of optical model differs from zero even for single-channel case!

Time scales in the statistical model:

Energy dependence of $\langle S_{ab}(E) \rangle$ realistically given by passage time of projectile through target. For $\langle S_{ab}(E) \rangle$ to be independent of energy,

that time must be very short in comparison with decay time of compound nucleus.

GOE is an equilibrium model (all states are coupled to each other). More realistic: A hierarchy of states of increasing complexity (2p 1h states, 3p 2h states etc. etc.). Thus GOE model realistic only if decay time >> equilibration time (about h / (several MeV)).

Statistical model is limited to energies up to 10 or 20 MeV above neutron threshold. At higher energies, preequilibrium reactions are important.

Treatment of direct reactions.

Case with direct reactions (non-diagonal matrix $<S^{dir}_{ab}>$) reduced exactly to case without direct reactions ($<S_{ab}>$ diagonal).

Matrix $P_{ab} = \delta_{ab} - \sum_c <S^{dir}_{ac}> <S^{dir}_{bc}>^*$ is hermitian. Diagonalized by energy-independent unitary transformation U. Then $(US^{dir}U^T)_{ab}$ is identical to S-matrix S_{ab} without direct reactions ($<S_{ab}> \propto \delta_{ab}$),

$$(US^{dir}U^T)_{ab} = S_{ab} = \delta_{ab} - 2i\pi \sum_{\mu\nu} W_{a\mu} D^{-1}_{\mu\nu} W_{\nu b} ,$$
$$D_{\mu\nu}(E) = E\, \delta_{\mu\nu} - (H^{GOE})_{\mu\nu} + i\pi \sum_c W_{\mu c} W_{c\nu} .$$

With $UX \rightarrow W$, $W_{a\mu}$ is real. Moreover, $\sum_\mu W_{a\mu} W_{b\mu} \propto \delta_{ab}$. Cross section $\propto |(USU^T)_{ab}|^2$ given in terms of S and U.

R. G. Satchler, Phys. Lett. 7 (1963) 55
Z. Vager, Phys. Lett. B 36 (1971) 269
M. Kawai, A. K. Kerman, K. W. McVoy, Ann. Phys. (N.Y.) 75 (1973) 156
C. Engelbrecht and H. A. Weidenmüller, Phys. Rev. C 8 (1973) 859
H. Nishioka and H. A. Weidenmüller, Phys. Lett. B 157 (1985) 101

4. Compound Nucleus Cross Section

Input: The diagonal elements $\langle S \rangle_{aa}$ or the "transmission coefficients"

$$T_a = 1 - |\langle S \rangle_{aa}|^2.$$

T_a (unitarity deficit of $\langle S_{aa} \rangle$) gives the probability of compound-nucleus formation in channel a. Number of input parameters = number of open channels.

Parameters of the model: Matrix elements $W_{\mu a}$ and parameter λ?

Total number = 1 + (N times number of channels). Is model underdetermined? NO!!! H^{GOE} is orthogonally invariant, so all moments of S can only depend on orthogonal invariants formed from the $W_{\mu a}$. The only such invariants are the bilinear forms $\sum_\mu W_{a\mu} W_{b\mu} \propto \delta_{ab}$. Moreover, S is dimensionless; so all moments can depend only on the dimensionless ratios $\sum_\mu (W_{a\mu})^2 / \lambda$. (The only way in which GOE parameter λ enters).

Parameters of the model: The dimensionless ratios $\sum_\mu (W_{a\mu})^2 / \lambda$.

Number of parameters = number of open channels: The model is well defined!

Calculate moments of $S_{ab}(E)$ by averaging over H^{GOE}. Difficulty: $N(N+1)/2$ Gaussian random variables appear in denominator of $S_{ab}(E)$.

Approaches:

(ι) $\Gamma \ll d$: Perturbation expansion and attempts to go beyond. Use Wigner-Dyson distribution of eigenvalues of $H_{\mu\nu}$ and Porter-Thomas distribution of partial widths.

A. M. Lane and J. E. Lynn, Proc. Phys. Soc. London LXX 8-A (1957) 557
P. A. Moldauer, Phys. Rev. 123 (1961) 968; ibid. 129 (1963) 754; ibid, 135 (1964) B 642; ibid. 136 (1964) B 947; ibid. 157 (1967) 907; ibid. 171 (1968) 1164; ibid. C 11 (1975) 426; ibid. C 12 (1975) 744; Rev. Mod. Phys. 36 (1964) 1079; Phys. Rev. Lett. 18 (1967) 249; Nucl. Phys. A 344 (1980) 185.
G. Reffo, F. Fabbri, H. Gruppelaar, Lett. Nuov. Cim. 17 (1976) 1

$$\langle |S^{fl}_{ab}|^2 \rangle = T_a \, T_b \, (\Sigma_c \, T_c)^{-1} \, W_{ab}$$

W_{ab} is "width fluctuation correction". $W = 1$ for single channel. $W_{aa} = 3$ for more than one channel and isolated resonances. Parametrizations for W_{ab} available.

Moldauer tried to go beyond $\Gamma \ll d$ using pole expansion of S-matrix. But pole parameters (positions and residues) linked by unitarity. Unresolved difficulty.

(ii) $\Gamma >> d$: Theories based on average unitarity or using an asymptotic expansion in d / Γ .

M. Kawai, A. K. Kerman, K. W. McVoy, Ann. Phys. (N.Y.) 75 (1973) 156
D. Agassi and H. A. Weidenmüller, Phys. Lett. B 56 (1975) 305
D. Agassi, H. A. Weidenmüller, G. Mantzouranis, Phys. Lett. C 22 (1975) 145
H. A. Weidenmüller, Ann. Phys. (N.Y.) 158 (1984) 120

$$<|S^{fl}_{ab}|^2> = (1 + \delta_{a\,b})\, T_a\, T_b\, (\sum_c T_c)^{-1} .$$

Width fluctuation correction equals two, in keeping with Vager and experimental data. Corrections from systematic expansion in inverse powers of $(d / \Gamma) \propto (\sum_c T_c)^{-1}$.

Elements of scattering matrix have Gaussian distribution. Second moment yields for normalized cross-section autocorrelation function the value ($\Gamma = (d/(2\pi)) \sum_c T_c$)

$$< |S^{fl}_{a\,b}(E + \varepsilon)|^2\, |S^{fl}_{a\,b}(E)|^2> / < |S^{fl}_{a\,b}(E)|^4> = 1 / (1 + (\varepsilon/\Gamma)^2) .$$

Complete derivation of Ericson’s model for cross-section fluctuations. Predicts exponential decay in time of compound nucleus.

(iii) Fit formulas with parameters determined by numerical simulation.

J. W. Tepel, H. M. Hofmann, H. A. Weidenmüller, Phys. Lett. B 49 (1974) 1
H. W. Hofmann, J. Richert, J. W. Tepel, H. A. Weidenmüller, Ann. Phys. (N.Y.) 90 (1975) 403
H. W. Hofmann, J. Richert, J. W. Tepel, Ann. Phys. (N.Y.) 90 (1975) 391
P. A. Moldauer, Phys. Rev. papers as cited above

Assume that cross section factorizes,

$$<|S^{fl}_{ab}(E)|^2> = V_a V_b (\sum_c V_c)^{-1} [1 + \delta_{ab} (R_a - 1)]$$

so that unitarity yields

$$T_a = V_a + (V_a)^2 (\sum_c V_c)^{-1} (R_a - 1) ,$$

and use the R_a as fit parameters.

Reasonable fits obtained for $R_a = 1 + 2/(1 + T_a^{1/2})$ but better parametrizations available. Similarly for $<S^{fl}_{aa} (S^{fl}_{bb})^*>$ (needed for direct reactions).

(iv) Exact result for two-point function

J. J. M. Verbaarschot, H. A. Weidenmüller, M. R. Zirnbauer, Phys. Lett. 149 B (1984) 263 and Phys. Rep. 129 (1985) 367
J. J. M. Verbaarschot, Ann. Phys. 168 (1986) 368

Obtain exact expression for $<S^{fl}_{ab}(E+\varepsilon)\,(S^{fl}_{cd}(E))^{*}>$ with help of "supersymmetry" integration technique (Efetov):

$<S^{fl}_{ab}(E+\varepsilon)\,(S^{fl}_{cd}(E))^{*}>$ = threefold integration involving T_c's and $<S_{aa}>$, $<S^{*}_{cc}>$.

(1) Limit of almost isolated resonances: Agreement with Moldauer.
(2) $\Gamma >> d$: Agreement with asymptotic expansion formula (including higher-order terms).
(3) Results coincide (within statistical errors) with fit formulas by Hofmann et al.
(4) For 2 channels and several choices of transmission coefficients, results agree numerically with those of maximum entropy approach (next transparency).

The problem is partly solved but … nearly impossible to calculate higher moments of S^{fl}!

(v) Maximum Entropy Approach

P. A. Mello, P. Pereyra, T. H. Seligman, Ann. Phys. (N.Y.) 161 (1985) 254
W. A. Friedman and P. A. Mello, Ann. Phys. (N.Y.) 161 (1985) 254

Use unitarity, symmetry, ergodicity, causality ($\langle S^k \rangle = \langle S \rangle^k$) of $S_{ab}(E)$ and a maximum entropy approach to derive complete probability distribution of S-matrix elements. The probability density **P**(S) is

$$\mathbf{P}(S) \propto [\det(1 - \langle S \rangle \langle S^* \rangle)]^{(n+1)/2} / |\det(1 - \langle S \rangle S^*)|^{n+1} .$$

In Ericson regime, this yields Hauser-Feshbach formula and Gaussian distribution of S-matrix elements.

Strong indications that this is the correct formula. So problem is solved completely?
Drawbacks: (i) Integration kernel is extremely difficult to handle.
(ii) The approach does not yield information about energy correlation of S-matrix elements.

5. Summary. Solved and Unsolved Problems

Theory: Consistent results from different approaches. Complete results for $<|S_{ab}|^2>$. Full theoretical understanding of Ericson fluctuations. Cross-section correlation functions missing for $\Gamma \approx d$.

Comparison with experiment: Very thorough corroboration for $\Gamma << d$ and in Ericson regime. None for $\Gamma \approx d$. But test using microwave billiards under way (Darmstadt).

Extensions: Isobaric analog resonances, isospin violation for $\Gamma >> d$, parity violation in epithermal neutron scattering.

Preequilibrium reactions: Require introduction of hierarchies of states with additional parameters. Conceptual simplicity of compound-nucleus model lost.

Compound-nucleus scattering is very general problem, especially in Ericson regime: Appears in passage of light through disordered media, transmission of electrons through mesoscopic devices, etc.